Ex.1. *Find the equation to the hyperbola with foci* (6, 4) *and* (−4, 4) *and eccentricity* = 2. **[M.U. 72S]**

Soln. The distance between the foci $= \sqrt{\{(6+4)^2 + (4-4)^2\}} = \sqrt{10^2} = 10.$

Thus $2ae = 10 \Rightarrow 2a \cdot 2 = 10 \quad \therefore \quad a = 5/2.$

Therefore $b^2 = a^2(e^2 - 1) = \dfrac{25}{4}(4-1) = \dfrac{25}{4} \times 3 = \dfrac{75}{4}.$

Hence the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \qquad \frac{4x^2}{25} - \frac{4y^2}{75} = 1$$

$$\Rightarrow \qquad 12x^2 - 4y^2 = 75.$$

Ex.2. *Find the centre and eccentricity of the hyperbola*

$$x^2 - 4y^2 - 2x + 24y - 37 = 0.$$

Soln. The given equation can be written as

$$(x^2 - 2x) - 4(y^2 - 6y) = 37$$

$$\Rightarrow \quad (x-1)^2 - 1 - 4\{(y-3)^2 - 9\} = 37$$

$$\Rightarrow \quad (x-1)^2 - 4(y-3)^2 = 2.$$

Writing $x - 1 = X$ and $y - 3 = Y$, we get $X^2 - 4Y^2 = 2$

which represents a hyperbola $\dfrac{X^2}{2} - \dfrac{Y^2}{1/2} = 1.$

The centre is given by $X = 0,\ Y = 0$ i.e. by $x = 1$ and $y = 3.$

Theeccentricityisgivenby

$$b^2 = a^2(e^2 - 1) \text{ i.e. by } \frac{1}{2} = 2(e^2 - 1)$$

which $\Rightarrow$ $e^2 - 4 = \dfrac{1}{4} \Rightarrow e^2 = 4 + \dfrac{1}{4} = \dfrac{17}{4} \quad \therefore \quad e = \sqrt{17}/2.$

Ex.3. *If* e_1 *and* e_2 *be the eccentricities of the hyperbolas*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1,$$

show that $\frac{1}{e_1^{\,2}} + \frac{1}{e_2^{\,2}} = 1.$

[B.U. 72H; M.U. 78A, 90; R.U. 68S, 70A, 7

Soln. From the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$

we have, $b^2 = a^2(e_1^2 - 1) \Rightarrow a^2e_1^2 = a^2 + b^2$ ...(1)

The second equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ can be written as $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

Hence $a^2 = b^2(e_2^2 - 1) \Rightarrow b^2e_2^2 = a^2 + b^2$ ...(2)

(1) $\Rightarrow$ $$\frac{1}{e_1^2} = \frac{a^2}{a^2 + b^2}.$$

(2) $\Rightarrow$ $$\frac{1}{e_2^2} = \frac{b^2}{a^2 + b^2}.$$

On adding, we get $\frac{1}{e_1^2} + \frac{1}{e_2^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1.$

Ex.4. *Chords of the circle $x^2 + y^2 = a^2$ touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Prove that their middle points lie on the curve*

$$(x^2 + y^2)^2 = a^2x^2 - b^2y^2.$$

[P.U. 71A; M.U. 71A, 92H]

Soln. The equation of the chord of the circle $x^2 + y^2 = a^2$ whose middle point is (α, β) is

$$x\alpha + y\beta = \alpha^2 + \beta^2 \qquad ...(1)$$

This is obtained by using the formula $S_1 = T$.

Again the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Hence, putting $x_1 = a\sec\theta$ and $y_1 = b\tan\theta$, the equation of the tangent to the hyperbola at the point 'θ' is

$$\frac{x \cdot a\sec\theta}{a^2} - \frac{y \cdot b\tan\theta}{b^2} = 1 \Rightarrow \frac{x\sec\theta}{a} - \frac{y}{b}\tan\theta = 1 \qquad ...(2)$$

Since (1) and (2) are the same, therefore

$$\frac{\alpha a}{\sec\theta} = \frac{\beta b}{-\tan\theta} = \frac{\alpha^2 + \beta^2}{1}$$

$$\Rightarrow \quad \sec\theta = \frac{a\alpha}{\alpha^2 + \beta^2} \text{ and } \tan\theta = -\frac{\beta b}{\alpha^2 + \beta^2}$$

$$\Rightarrow \quad \sec^2\theta - \tan^2\theta = \frac{a^2\alpha^2}{(\alpha^2 + \beta^2)^2} - \frac{\beta^2 b^2}{(\alpha^2 + \beta^2)^2}$$

$$\Rightarrow \quad \frac{a^2\alpha^2 - \beta^2 b^2}{(\alpha^2 + \beta^2)^2} = 1 \Rightarrow (\alpha^2 + \beta^2)^2 = a^2\alpha^2 - \beta^2 b^2$$

Hence the locus of the middle point is $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$.

Ex.5. *Prove that the locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is $(y^2 - x^2)^3 = 4a^2x^2y^2$.*

Soln. The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point

$$(x_1, y_1) \text{ is } \frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{-y_1/b^2}.$$

In our case, the equation of the normal to the hyperbola $x^2 - y^2 = a^2$ at the point $(a \sec\theta, a \tan\theta)$ is

$$\frac{x - a\sec\theta}{\sec\theta/a} = \frac{y - a\tan\theta}{-\tan\theta/a}$$

$$\Rightarrow \quad \frac{x}{\sec\theta} - a = -\frac{y}{\tan\theta} + a \Rightarrow \frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a. \qquad ...(1)$$

Again, the equation to the chord of the given hyperbola whose middle point is (α, β) is

$$x\alpha - y\beta = \alpha^2 - \beta^2. \qquad ...(2)$$

This is obtained from the formula $S_1 = T$.

Since (1) and (2) represent the same line, therefore comparing (1) and (2), we get

$$\alpha\sec\theta = -\beta\tan\theta = \frac{\alpha^2 - \beta^2}{2a}$$

$$\Rightarrow \quad \sec\theta = \frac{\alpha^2 - \beta^2}{2a\alpha} \text{ and } \tan\theta = -\frac{\alpha^2 - \beta^2}{-2a\beta}$$

$$\Rightarrow \quad \sec^2\theta - \tan^2\theta = \frac{(\alpha^2 - \beta^2)^2}{4a^2\alpha^2} - \frac{(\alpha^2 - \beta^2)^2}{4a^2\beta^2}$$

$$\Rightarrow \quad 1 = \frac{(\alpha^2 - \beta^2)^2}{4a^2}\left\{\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right\} \Rightarrow 4a^2\alpha^2\beta^2 = (\alpha^2 - \beta^2)^2(\beta^2 - \alpha^2)$$

$$\Rightarrow \quad (\beta^2 - \alpha^2)^3 = 4a^2\alpha^2\beta^2.$$

Hence the locus of (α, β) is $(y^2 - x^2)^3 = 4a^2x^2y^2$.